Exercise 19

Use the method of undetermined coefficients to find the particular solution for the following initial value problems:

$$u'' - u = -2\sin x, \quad u(0) = 1,\ u'(0) = 2$$

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' - u_c = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \to \quad u_c' = re^{rx} \quad \to \quad u_c'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 1 = 0$$

Factor the left side.

$$(r+1)(r-1) = 0$$

$r = -1$ or $r = 1$, so the complementary solution is

$$u_c(x) = C_1 e^{-x} + C_2 e^{x}.$$

We can write this in terms of hyperbolic sine and hyperbolic cosine.

$$u_c(x) = A\cosh x + B\sinh x$$

Now we turn our attention to the particular solution. Because the inhomogeneous term is $-2\sin x$ and u' is not present on the left side, try a particular solution of the form, $u_p = C\sin x$. Plugging this form into the ODE yields $-C\sin x - C\sin x = -2\sin x$, which means $C = 1$. Thus, $u_p = \sin x$. Therefore, the general solution to the ODE is

$$u(x) = A\cosh x + B\sinh x + \sin x.$$

These constants can be determined since initial conditions are given.

$$u'(x) = A\sinh x + B\cosh x + \cos x$$

$$u(0) = A = 1$$
$$u'(0) = B + 1 = 2$$

The solution to this system of equations is $A = 1$ and $B = 1$. Therefore,

$$u(x) = \cosh x + \sinh x + \sin x.$$

We can check that this is the solution. The first and second derivatives are

$$u' = \sinh x + \cosh x + \cos x$$
$$u'' = \cosh x + \sinh x - \sin x.$$

Hence,

$$u'' - u = \cancel{\cosh x} + \cancel{\sinh x} - \sin x - (\cancel{\cosh x} + \cancel{\sinh x} + \sin x) = -2 \sin x,$$

which means this is the correct solution.